TRANSIENT TEMPERATURE FILED IN LAMINATED ORTHOTROPIC CYLINDER

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UDC 536.12:539.377

A method based on the Green function is proposed for solving the problem of thermal conductivity in a finite, laminated orthotropic cylinder. The temperature field in a three-layer cylindrical wall containing a thermal insulation layer characterized by low thermal conductivity is calculated.

Consider a finite, laminated hollow cylinder with a circular transverse cross section, composed of an arbitrary number of concentric orthotropic layers with different thermophysical characteristics. We assume that perfect thermal contact exists between the cylinder layers and that heat exchange with the ambient, which is at an assigned constant temperature, occurs at the surfaces bounding the cylinder. The initial cyclinder temperature is equal to t_0.

We represent the thermophysical characteristics of this multilayer cylinder as a single whole in the following form [1]:

$$p(r) = p_1 + \sum_{j=1}^{n-1} (p_{j+1} - p_j) S(r - r_j), \tag{1}$$

where p_j is the characteristic of the j-th layer, and n is the number of layers.

For determining the transient temperature field, we use the equation [1]

$$Lt = c_v(r) \frac{\partial t}{\partial \tau}, \tag{2}$$

the conditions for perfect thermal contact

$$t|_{r=r_j+0} = t|_{r=r_j-0}, \quad \left.\frac{\partial t}{\partial r}\right|_{r=r_j+0} = K_j^\lambda \left.\frac{\partial t}{\partial r}\right|_{r=r_j-0} \tag{3}$$

and the boundary conditions

$$\left.\left[\frac{\partial t}{\partial r} - h_1(t - t_{\text{me}}^{(1)})\right]\right|_{r=r_0} = 0, \quad \left.\left[\frac{\partial t}{\partial r} + h_2(t - t_{\text{me}}^{(2)})\right]\right|_{r=r_n} = 0, \tag{4}$$

$$\left.\left[\frac{\partial t}{\partial z} - h_3(t - t_{\text{me}}^{(3)})\right]\right|_{z=0} = 0, \quad \left.\left[\frac{\partial t}{\partial z} + h_4(t - t_{\text{me}}^{(4)})\right]\right|_{z=H} = 0, \tag{5}$$

$$t|_{\tau=0} = t_0,$$

where

$$L = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda_r(r)\frac{\partial}{\partial r}\right) + \lambda_z(r)\frac{\partial^2}{\partial z^2}; \quad K_j^\lambda = \frac{\lambda_r^{(j)}}{\lambda_r^{(j+1)}}.$$

*Deceased.

Institute of Problems in Applied Mechanics and Mathematics, Academy of Sciences of the Ukraine, L'vov. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 62, No. 2, pp. 325-330, February, 1992. Original article submitted June 4, 1991.

Using Green's function, we represent the solution of problem (2)-(5) in the following form [2]:

$$t(r, z, \tau) = t_0 + \int_0^{\tau} \Big\{ \int_0^{H} [r_0 \lambda_r^{(1)} h_1 T_{\text{me}}^{(1)} G|_{\rho=r_0} +$$

$$+ r_n \lambda_r^{(n)} h_2 T_{\text{me}}^{(2)} G|_{\rho=r_n}]\, d\zeta + \int_{r_0}^{r_n} \rho \lambda_z(\rho)\, [h_3 T_{\text{me}}^{(3)} G|_{\zeta=0} + \tag{6}$$

$$+ h_4 T_{\text{me}}^{(4)} G|_{\zeta=H}]\, d\rho \Big\}\, d\tau^0.$$

Here, $T_{\text{me}}^{(i)} = t_{\text{me}}^{(i)} - t_0$, while the Green function $G(\rho, \zeta; r, z; \tau')(\tau' = \tau - \tau^0)$ is determined by means of the equation

$$LG = c_v(r) \frac{\partial G}{\partial \tau'} . \tag{7}$$

The contact conditions are given by

$$G|_{r=r_j+0} = G|_{r=r_j-0}, \quad \frac{\partial G}{\partial r}\bigg|_{r=r_j+0} = K_j^{\lambda} \frac{\partial G}{\partial r}\bigg|_{r=r_j-0} \tag{8}$$

while the boundary conditions are defined by

$$\left[\frac{\partial G}{\partial r} - h_1 G\right]\bigg|_{r=r_0} = 0, \quad \left[\frac{\partial G}{\partial r} + h_2 G\right]\bigg|_{r=r_n} = 0, \tag{9}$$

$$\left[\frac{\partial G}{\partial z} - h_3 G\right]\bigg|_{z=0} = 0, \quad \left[\frac{\partial G}{\partial z} + h_4 G\right]\bigg|_{z=H} = 0, \tag{10}$$

$$G|_{\tau'=0} = \frac{1}{\rho c_v(\rho)} \delta(r-\rho)\, \delta(z-\zeta).$$

In order to find the Green function, we apply to (7)-(10) the Fourier and Laplace integral transforms with respect to the variables z and τ [3]. As a result, we obtain the ordinary differential equation

$$\bar{L}\bar{G} = -\frac{1}{\rho} \Phi_l(\zeta)\, \delta(r-\rho) \tag{11}$$

for the conditions

$$\bar{G}|_{r=r_j+0} = \bar{G}|_{r=r_j-0}, \quad \frac{d\bar{G}}{\partial r}\bigg|_{r=r_j+0} = K_j^{\lambda} \frac{d\bar{G}}{dr}\bigg|_{r=r_j-0}, \tag{12}$$

$$\left[\frac{d\bar{G}}{dr} - h_1 \bar{G}\right]\bigg|_{r=r_0} = 0, \quad \left[\frac{d\bar{G}}{dr} + h_2 \bar{G}\right]\bigg|_{r=r_n} = 0, \tag{13}$$

where

$$\bar{L} = \frac{1}{r} \frac{d}{dr}\left(r \lambda_r(r) \frac{d}{dr}\right) - \lambda_z(r) \beta_l^2 - c_v(r)\, s;$$

$\Phi_l(\zeta) = \beta_l \cos \beta_l \zeta + h_3 \sin \beta_l \zeta$, s is the Laplace transform parameter, and $\beta_l\ (l = 1, 2, \ldots)$ are the positive roots of the transcendental equation

$$(\beta^2 - h_3 h_4)\sin\beta H - \beta(h_3 + h_4)\cos\beta H = 0.$$

Solving by means of the method presented in [4] the homogeneous equation (11) for conditions (12), we find

$$\bar{G}_0 = c_1 f_1(r) + c_2 f_2(r),$$

where

$$f_i(r) = \sum_{k=1}^{2} B^{(i,k)}(r) Z_0^{(k)}(\varepsilon^*(r)r);\quad B_{j+1}^{(i,k)} = \sum_{m=1}^{2} B_j^{(i,m)} P_{k\pm 1,m}^{(j+1)};$$

$$P_{k,m}^{(j+1)} = (-1)^k \frac{\pi r_j}{2}[\varepsilon_{j+1}^* Z_1^{(k)}(\varepsilon_{j+1}^* r_j) Z_0^{(m)}(\varepsilon_j^* r_j) - K_j^{\lambda}\varepsilon_j^* Z_0^{(k)}(\varepsilon_{j+1}^* r_j)\times$$

$$\times Z_1^{(m)}(\varepsilon_j^* r_j)];\quad \varepsilon^*(r) = \sqrt{-(s/a(r) + \beta_l^2\gamma(r))};$$

$$\gamma(r) = \frac{\lambda_z(r)}{\lambda_r(r)};\quad B_1^{(i,k)} = \delta_{ik};\quad k \pm 1 = \begin{cases} 2, & \text{if } k = 1, \\ 1, & \text{if } k = 2; \end{cases}$$

c_i are the integration constants, $\varepsilon^*(r)$ and $B^{(i,k)}(r)$ are represented in the form given by (1), $Z_i^{(1)}(x) = Y_i(x)$ and $Z_i^{(2)}(x) = J_i(x)$.

The solution of the nonhomogeneous equation (11), obtained by variation of the constants, assumes the following form after boundary conditions (13) are satisfied:

$$\bar{G} = \frac{\pi\Phi_l(\zeta)}{2\lambda_r^{(1)} A_l}[v_1(\rho)v_2(r)S(r-\rho) + v_1(r)v_2(\rho)S(\rho - r)].$$

Here,

$$v_1(r) = V_0^{(1)} f_2(r) - V_0^{(2)} f_1(r);\quad v_2(r) = V_n^{(1)} f_2(r) - V_n^{(2)} f_1(r);$$

$$A_l = V_0^{(1)} V_n^{(2)} - V_0^{(2)} V_n^{(1)};\quad V_0^{(i)} = -h_1 Z_0^{(i)}(\varepsilon_1^* r_0) - \varepsilon_1^* Z_1^{(i)}(\varepsilon_1^* r_0);$$

$$V_n^{(i)} = \sum_{k=1}^{2} B_n^{(i,k)} M_{k,n};\quad M_{k,n} = h_2 Z_0^{(k)}(\varepsilon_n^* r_n) - \varepsilon_n^* Z_1^{(k)}(\varepsilon_n^* r_n).$$

Passing to inverse transforms and using the expansion theorem of operational calculus, we obtain the final expression for the function after a number of transformations:

$$G = \frac{2\pi}{\lambda_r^{(1)}} \sum_{l=1}^{\infty} \frac{1}{N_l}\Phi_l(\zeta)\Phi_l(z) \sum_{m=1}^{\infty}\left\{\frac{1}{E}\exp(-\mu^2\tau')\times\right. \tag{14}$$

$$\left.\times[v_1(\rho)v_2(r)S(r-\rho) + v_1(r)v_2(\rho)S(\rho-r)]\right\}\Bigg|_{\mu=\mu_{l,m}},$$

where

$$N_l = h_3 + (\beta_l^2 + h_3^2)(H + h_4/(\beta_l^2 + h_4^2));$$

$$v_1(r) = \frac{\pi r_{j-1}}{2}[\psi_{1,0}(\varepsilon_j, r_{j-1}, r)d_{j-1}^{(1)} + K_{j-1}^{\lambda}\psi_{0,0}(\varepsilon_j, r, r_{j-1})\times$$

$$\times d_{j-1}^{(2)}],\quad \text{if}\quad r_{j-1} \leqslant r \leqslant r_j\ (j = 2, 3, \ldots, n);$$

$$v_1(r) = -h_1\psi_{0,0}(\varepsilon_1, r, r_0) + \psi_{1,0}(\varepsilon_1, r_0, r),\quad \text{if}\quad r_0 \leqslant r \leqslant r_1;$$

$$v_2(r) = -h_2 F_{n,j}^{(1)}(r) + F_{n,j}^{(2)}(r),\quad \text{if}\quad r_{j-1} \leqslant r \leqslant r_j\ (j = 1, \ldots, n);$$

$$F_{i,j}^{(k+1)}(r) = \frac{\pi r_{i-1}}{2}[b_i^k \psi_{1,k}(\varepsilon_i, r_{i-1}, r_i) F_{i-1,j}^{(1)}(r) + K_{i-1}^{\lambda}\psi_{k,0}(\varepsilon_i, r_i, r_{i-1})\times$$

$$\times F_{i-1,j}^{(2)}(r)]\ (i = j+1, \ldots, n);\quad F_{j,j}^{(k+1)}(r) = \frac{\lambda_r^{(1)}}{\lambda_r^{(j)}}\psi_{k,0}(\varepsilon_j, r_j, r);$$

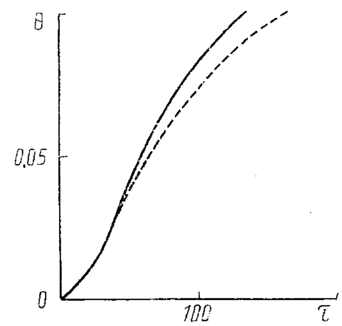

Fig. 1. Maximum dimensionless temperature of the third layer $\theta = t/t_c^{(1)}$ as a function of τ (sec) in the case of an orthotropic (solid curve) and isotropic (dashed curve) first layer.

$$A_l = h_2 d_n^{(1)} - d_n^{(2)};\ E = -h_2\omega_n^{(1)} + \omega_n^{(2)};\ \omega_1^{(k+1)} = -\frac{1}{a_1}(h_1\eta_1^{(k+1)} b_1^{k-1} + \eta_1^{(k+3)})$$

$$d_1^{(k+1)} = -h_1\psi_{k,0}(\varepsilon_1, r_1, r_0) + b_1^k \psi_{1,k}(\varepsilon_1, r_0, r_1);$$

$$\begin{Bmatrix} d_i^{(k+1)} \\ \omega_i^{(k+1)} \end{Bmatrix} = \frac{\pi r_{i-1}}{2}\left[b_i^k \psi_{1,k}(\varepsilon_i, r_{i-1}, r_i) \begin{Bmatrix} d_{i-1}^{(1)} \\ \omega_{i-1}^{(1)} \end{Bmatrix} + \right.$$

$$+ K_{i-1}^{\lambda}\psi_{k,0}(\varepsilon_i, r_i, r_{i-1}) \begin{Bmatrix} d_{i-1}^{(2)} \\ \omega_{i-1}^{(2)} \end{Bmatrix} - \frac{\eta_i^{(k+3)}}{a_i} \begin{Bmatrix} 0 \\ d_{i-1}^{(1)} \end{Bmatrix} +$$

$$\left. + K_{i-1}^{\lambda} \frac{\eta_i^{(k+1)}}{a_i} b_i^{k-1} \begin{Bmatrix} 0 \\ d_{i-1}^{(2)} \end{Bmatrix} \right] \ (i = 2, 3, \ldots, n),$$

$$\eta_i^{(k+1)} = r_{i-1}\psi_{1,k}(\varepsilon_i, r_{i-1}, r_i) + (-1)^{k+1} r_i \psi_{1-k,0}(\varepsilon_i, r_i, r_{i-1});$$

$$\eta_i^{(k+3)} = r_{i-1}\psi_{k,0}(\varepsilon_i, r_{i-1}, r_i) + (-1)^k r_i \psi_{1,1-k}(\varepsilon_i, r_i, r_{i-1})\ (k = 0, 1);$$

$$\psi_{\nu,i}(\varepsilon_j, x, y) = \varepsilon_j^{|\nu-i|}\{J_\nu(\varepsilon_j x) Y_i(\varepsilon_j y) - Y_\nu(\varepsilon_j x) J_i(\varepsilon_j y)\}, \quad \text{if} \quad b_j > 0;$$

$$\psi_{\nu,i}(\varepsilon_j, x, y) = -\frac{2}{\pi}(-\varepsilon_j)^{|\nu-i|}\{I_\nu(\varepsilon_j x) K_i(\varepsilon_j y) - (-1)^{\nu+i} K_\nu(\varepsilon_j x) I_i(\varepsilon_j y)\},$$

$$\text{if} \quad b_j < 0\ (\nu = 0, 1;\ i = 0, 1);\ \varepsilon_j = \sqrt{|b_j|};$$

$b_j = \mu^2/a_j - \beta_l^2\gamma_j$; $\mu_{\ell,m}$ are the positive roots of the transcendental equation $A_\ell = 0$.

From the above relationships and on the basis of the modified Bessel functions, it follows that $\mu_{\ell,m}$ belongs to the $(\beta_l D; +\infty)$ interval, since $A_\ell > 0$ for μ values from the $[0; \beta_l D]$ segment. Here $D = \min\{\sqrt{\gamma_j a_j}/j = 1, 2, \ldots, n\}$.

On the basis of (6) and (14), the temperature field of a laminated cylinder is determined by the expression

$$t = t_0 + \theta_1(r, z) + \theta_2(r, z, \tau), \tag{15}$$

where

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 4\sum_{l=1}^{\infty}\frac{\Phi_l(z)}{N_l}\sum_{m=1}^{\infty}\left[\begin{Bmatrix} 1 \\ -\exp(-\mu^2\tau) \end{Bmatrix}\frac{1}{E\mu^2}\times\Big((\sin\beta_l H + h_3(1-\cos\beta_l H)/\beta_l)(h_1 T_{me}^{(1)} v_2(r) + h_2 T_{me}^{(2)} v_1(r)) + \right.$$

$$\left.\left. + \frac{\pi}{2\lambda_r^{(1)}}(h_3 T_{me}^{(3)}\beta_l + h_4 T_{me}^{(4)}\Phi_l(H))(v_1(r)u_2(r) + v_2(r)u_1(r))\Big)\right]\right|_{\mu=\mu_{l,m}};$$

$$u_1(r)=\frac{2}{\pi}h_1q_1+\sum_{i=1}^{j-1}(q_i-K_i^{\lambda}q_{i+1})r_id_i^{(2)}+q_j\frac{\pi r_{j-1}}{2}r\times$$
$$\times[K_{j-1}^{\lambda}\psi_{1,0}(\varepsilon_j,r,r_{j-1})d_{j-1}^{(2)}+b_j\psi_{1,1}(\varepsilon_j,r_{j-1},r)d_{j-1}^{(1)}],$$
if $r_{j-1}\leqslant r\leqslant r_j\,(j=2,3,\ldots,n);\; q_i=\lambda_z^{(i)}/b_i;$
$$u_1(r)=q_1\left\{\frac{2}{\pi}h_1+r\,[b_1\psi_{1,1}(\varepsilon_1,r_0,r)-h_1\psi_{1,0}(\varepsilon_1,r,r_0)]\right\},\quad \text{if}\quad r_0\leqslant r\leqslant r_1;$$
$$u_2(r)=\sum_{p=1}^{2}(-1)^{p+1}h_2^{2-p}\left[R_{n,j}^{(p)}(r)+\sum_{i=j}^{n-1}(R_{n,i+1}^{(p)}(r_i)-\right.$$
$$\left.-R_{n,i}^{(p)}(r_i))\right],\quad \text{if}\quad r_{j-1}\leqslant r\leqslant r_j\,(j=1,2,\ldots,n);$$
$$R_{i,j}^{(k+1)}(r)=\frac{\pi r_{i-1}}{2}\,[b_i^k\,\psi_{1,k}(\varepsilon_i,r_{i-1},r_i)\,R_{i-1,j}^{(1)}(r)+$$
$$+K_{i-1}^{\lambda}\psi_{k,0}(\varepsilon_i,r_i,r_{i-1})\,R_{i-1,j}^{(2)}(r)]\,(i=j+1,\ldots,n;\;k=0,1);$$
$$R_{j,j}^{(1)}(r)=\frac{\lambda_r^{(1)}}{\lambda_r^{(j)}}q_j\left[\frac{2}{\pi}-r\psi_{1,0}(\varepsilon_j,r,r_j)\right];$$
$$R_{j,j}^{(2)}(r)=-\lambda_r^{(1)}\gamma_j r\psi_{1,1}(\varepsilon_j,r,r_j).$$

We represent in a simpler form the expression $\theta_1(r,z)$, which describes the steady-state temperature distribution

$$\theta_1=Q_1+Q_2z+2\sum_{l=1}^{\infty}\frac{\Phi_l(z)}{N_l}\left[\frac{h_2D_2v_1(r)+h_1D_1v_2(r)}{A_l}\right]\Bigg|_{\mu=0},\tag{16}$$

having solved the equation

$$L\theta_1=0$$

for contact conditions (3) and boundary condition (4). Here,

$$Q_1=\frac{T_{\text{me}}^{(3)}h_3(1+h_4H)+T_{\text{me}}^{(4)}h_4}{h_4+h_3(1+h_4H)};\quad Q_2=\frac{h_3h_4(T_{\text{me}}^{(4)}-T_{\text{me}}^{(3)})}{h_4+h_3(1+h_4H)};$$
$$D_i=(Q_1-T_{\text{me}}^{(i)})\left[\sin\beta_lH+\frac{h_3}{\beta_l}(1-\cos\beta_lH)\right]+$$
$$+Q_2\left[\left(H+\frac{h_3}{\beta_l^2}\right)\sin\beta_lH+\frac{(1-h_3H)\cos\beta_lH-1}{\beta_l}\right](i=1,2).$$

In order to find (16), we used the Green function of the steady-state problem,

$$G=\frac{\pi}{\lambda_r^{(1)}}\sum_{l=1}^{\infty}\frac{\Phi_l(z)\Phi_l(\zeta)}{N_l}\left\{\frac{1}{A_l}[v_1(\rho)v_2(r)S(r-\rho)+\right.$$
$$\left.+v_1(r)v_2(\rho)S(\rho-r)]\right\}\Bigg|_{\mu=0},$$

which satisfies the equation

$$LG=-\frac{1}{\rho}\,\delta(r-\rho)\,\delta(z-\zeta)$$

and conditions (8) and (9), which is obtained by using the same approach as in the case of the Green function of the transient problem.

Expressions (15) and (16) were used for calculating the temperature field in a three-layer cylindrical wall of a power plant for

$$K_1^\lambda = 1500;\ K_2^\lambda = 0.002;\ \gamma_1 = 2{,}14;\ \gamma_2 = \gamma_3 = 1;\ \frac{c_v^{(1)}}{c_v^{(2)}} = 5;$$

$$\frac{c_v^{(2)}}{c_v^{(3)}} = 0.1;\ \frac{r_0}{r_1} = 0{,}5;\ \frac{r_1}{r_2} = 0.91;\ \frac{r_2}{r_3} = 0.88;\ \frac{H}{r_3} = 2;$$

$$h_1 = 20;\ h_2 = 0{,}67;\ h_3 = 0;\ h_4 = \infty;\ t_0 = t_{me}^{(2)} = t_{me}^{(4)} = 0\text{K}.$$

We also considered the case where the first layer was isotropic and the thermal conductivity coefficient defined by the relationship

$$\lambda^{(1)} = \sqrt{\lambda_r^{(1)}\lambda_z^{(1)}}.$$

Figure 1 provides the calculation results regarding the time in which the temperature 0.1 $t_{me}^{(1)}$ is reached in the third layer. If the first layer is orthotropic the time in which this temperature reaches its maximum value at the point ($r = r_2$, $z = 0$) amounts to 136 sec, which is 28 sec less in comparison with the results of calculations without an allowance for orthotropy.

NOTATION

t, cylinder temperature; $\lambda_r^{(j)}$, $\lambda_z^{(j)}$, $c_v^{(j)}$, a_j, r_j, thermal conductivity coefficients in the radial and axial directions, volumetric specific heat, thermal diffusivity coefficient, and the outside radius of the j-th layer, respectively; r_0, radius of the inside surface of the cylinder; H, cylinder height; $t_{me}^{(i)}$ and h_i, temperature of the medium and mean value of the ratio of the heat-transfer coefficient to the normal component of the thermal conductivity coefficient at the corresponding surfaces (i = 1, 2, 3, and 4), respectively; S(x), Heaviside unit function; δ(x) Dirac's delta function; δ_{ik}, Kronecker's symbol; $J_i(x)$ and $Y_i(x)$, i-order Bessel functions; $I_i(x)$ and $K_i(x)$, modified i-order Bessel functions; r, φ, z, cylindrical coordinates; τ, time.

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